CS103 Handout 14
Fall 2013 October 21, 2013

Practice Midterm Exam 1

This exam is open-book and open-note. You may use a computer only to look at notes that you yourself have written, to access the course website and the tools there, and to read an online copy of one of the recommended readings. No other use of the computer is permitted during this exam. You must hand-write all of your solutions on this physical copy of the exam. No electronic submissions will be considered without prior consent of the course staff.

SUNetID: ______________________

Last Name: ______________________

First Name: ______________________

I accept both the letter and the spirit of the honor code. I have not received any unpermitted assistance on this test, nor will I give any. My answers are my own work and are not adapted from any unpermitted sources. I will not use a computer except in the ways specified at the top of the exam.

(signed) ______________________________

You have three hours to complete this midterm. There are 180 total points, and this midterm is worth 15% of your total grade in this course. You may find it useful to read through all the questions to get a sense of what this midterm contains. As a rough sense of the difficulty of each question, there is one point on this exam per minute of testing time.

Question		**Points**	**Grader**
(1) Mathematical Logic	(20)	/ 20	
(2) Finite Automata	(15)	/ 15	
(3) Induction	(45)	/ 45	
(4) Relations and Functions	(55)	/ 55	
(5) The Pigeonhole Principle	(45)	/ 45	
	(180)	**/ 180**	

Note: Normally, there would be a lot of whitespace on this exam so that you would have room to write answers. To conserve paper, we've removed it in this practice exam.

Problem One: Mathematical Logic (20 points total)

(i) Set Theory (10 Points)

Given the predicates

> *Set*(S), which states that S is a set, and
>
> $x \in S$, which states that x is an element of S,

Write a statement in first-order logic that states "for any x and y, there is a set containing just the elements x and y." This is called the ***axiom of pairing***. Your formula can use any constructs of first-order logic (quantifiers, connectives, equality, etc.), but you should not use any functions or constants and should only use the predicates given above.

(ii) Set Theory, Part Two (10 Points)

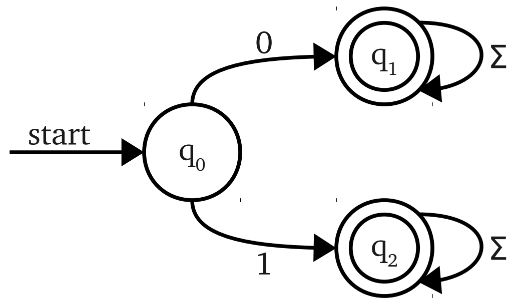
For any sets A and B, consider the set S defined below:

$$S = \{ x \mid \neg(x \in A \rightarrow x \in B) \}$$

Write an expression for S in terms of A and B using the standard set operators (union, intersection, etc.) and briefly justify why your answer is correct.

Problem Two: Finite Automata (15 Points)

Consider the following DFA, which we'll call *D*:



(i) Identifying the Alphabet (5 Points)

What is the alphabet of *D*? Briefly justify your answer (a sentence or two should be sufficient).

(ii) DFA Compression (10 Points)

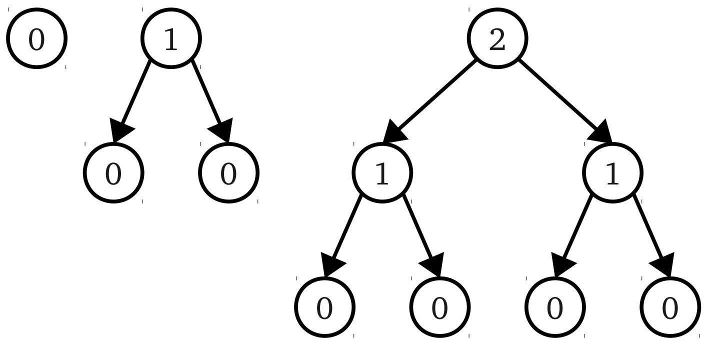
Construct a DFA with the same language as *D*, but which has fewer states.

Problem Three: Induction (45 points)

The *perfect binary trees* are special class of directed trees with numerous applications throughout computer science. The perfect binary trees are defined inductively:

- A perfect binary tree of order 0 is a single node.
- A perfect binary tree of order $n + 1$ is a single node with two children, both of which are perfect binary trees of order n.

For example, here are the first few perfect binary trees:



(i) Counting Nodes (15 Points)

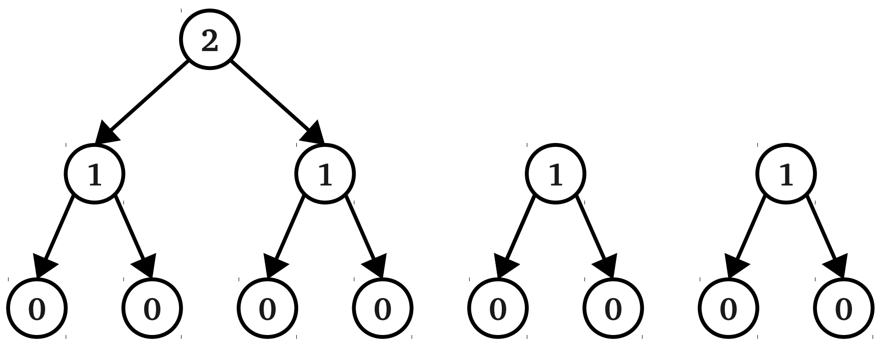
Prove that for any $n \in \mathbb{N}$, a perfect binary tree of order n has $2^{n+1} - 1$ nodes.

Given any collection of nodes, it is always possible to group those nodes into a collection of perfect binary trees such that

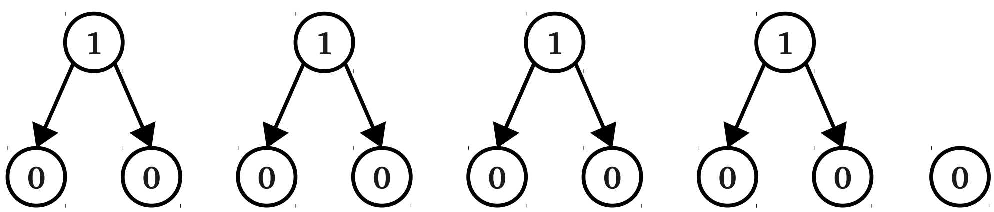
1. There are at most two trees of each order, and
2. If there are two trees of some order n, there are no trees of any order n' for any $n' < n$.

This last condition means that if there are two perfect binary trees of order n in the collection, then n has to be the smallest order used in the collection. A grouping of nodes into perfect binary trees this way is called a ***skew binary decomposition***.

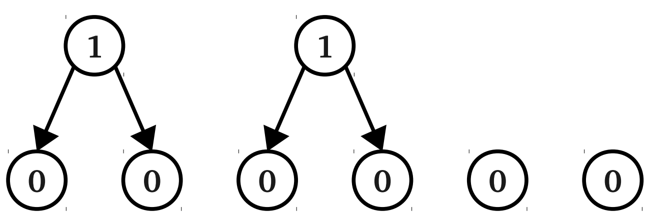
For example, a skew binary decomposition for thirteen nodes is shown here:



However, the following is not a skew binary decomposition for thirteen nodes, because there are more than two trees of order one:



Additionally, the following is not a skew binary decomposition for eight nodes, because there are two trees of order one, but there are also trees of a smaller order.



(ii) The Skew Binary Decomposition **(30 Points)**

Prove that for any $n \in \mathbb{N}$, any set of n nodes has a skew binary decomposition.

Problem Four: Relations and Functions (55 Points)

(i) Subsets and Cardinality (10 Points)

Prove, for all sets A and B, that if $A \subseteq B$, then $|A| \leq |B|$.

One of the major results from modern set theory is that there are some collections of objects that cannot be gathered into a set. Of particular importance is that there is no "set of all sets," meaning that the set U described below does not exist:

$$U = \{ S \mid S \text{ is a set} \}$$

(ii) The Set of all Sets (20 Points)

Using your result from part (i) and Cantor's Theorem, prove that U does not exist.

(A note: If U were to exist, it would have to contain itself. This isn't immediately a problem; in some versions of set theory, it is possible for a set to contain itself. You should not try to use this fact to show that U does not exist.)

Consider an arbitrary set A. There may be many different partial order relations over this set A. We'll denote by PO the set of all partial order relations over the set A.

Let's define a new relation R over PO as follows: for any partial orders $\leq_1, \leq_2 \in PO$, we have

$$\leq_1 R \leq_2 \quad \text{iff} \quad \text{for every } x, y \in A, \text{ if } x \leq_1 y, \text{ then } x \leq_2 y$$

(iii) Meta Partial Orders (25 Points)

Prove that R is a partial order over PO. (A note: given two relations R_1 and R_2 over some set A, $R_1 = R_2$ iff for any $x, y \in A$, we have that xR_1y iff xR_2y.)

Problem Five: The Pigeonhole Principle (45 points total)

Suppose that you have a sequence of distinct real numbers $S = \langle x_1, x_2, \ldots, x_n \rangle$. A ***subsequence*** of S is a sequence of elements from S where the elements appear in the same relative order as they do in S. For example, consider the sequence S defined as follows:

$$S = \langle 1, 2, 3, 4, 5, 6 \rangle$$

Then $\langle 1, 3, 6 \rangle$ is a subsequence of S, as is $\langle 2, 3, 4, 5 \rangle$. However, $\langle 3, 2, 1 \rangle$ is not a subsequence of S (its elements are not in the same relative order), nor is $\langle 1, 4, 7 \rangle$ (since 7 is not an element of S).

Given a sequence S, an ***ascending subsequence*** of S is a subsequence T where each element of T is strictly greater than all previous elements of T. A ***descending subsequence*** of S is a subsequence T where each element of T is strictly smaller than all previous elements of T. For example, given the sequence

$$S = \langle 106, 103, 107, 109, 110, 161 \rangle$$

The sequence $\langle 106, 103 \rangle$ is a descending subsequence of S, and the sequence $\langle 103, 109, 161 \rangle$ is an ascending subsequence of S. The sequence $\langle 107 \rangle$ is both an ascending and descending subsequence of S.

In this problem, you will prove a result called the ***Erdős–Szekeres Theorem***:

Any sequence of $rs + 1$ distinct real numbers contains an ascending subsequence of length $r + 1$ or a descending subsequence of length $s + 1$ (or both).

Suppose that our sequence of distinct real numbers is $S = \langle x_1, x_2, \ldots, x_{rs+1} \rangle$. Let's associate with each element x_k of this sequence a pair of natural numbers (I_k, D_k) with the following meaning:

I_k is the length of the longest ***increasing*** subsequence of S that ends at position k.

D_k is the length of the longest ***decreasing*** subsequence of S that ends at position k.

For example, consider the sequence $\langle 40, 20, 10, 30, 50 \rangle$. Then

$$(I_1, D_1) = (1, 1)$$

$$(I_2, D_2) = (1, 2)$$

$$(I_3, D_3) = (1, 3)$$

$$(I_4, D_4) = (2, 2)$$

$$(I_5, D_5) = (3, 1)$$

You might want to take a minute to check why these values are correct.

Feel free to tear this page out as a reference.

(i) Measuring Subsequences **(5 Points)**

Let k be an arbitrary natural number where $1 \leq k \leq rs + 1$. Prove that $I_k \geq 1$ and $D_k \geq 1$.

(ii) Distinguishing Pairs **(10 Points)**

Let j and k be arbitrary natural numbers where $1 \leq j \leq rs + 1$ and $1 \leq k \leq rs + 1$. Prove that if $j \neq k$, then $(I_j, D_j) \neq (I_k, D_k)$. To keep your proof short, we recommend assuming without loss of generality that $j < k$.

(iii) Putting Everything Together **(30 Points)**

Using your results from parts (i) and (ii), prove that any sequence of $rs + 1$ distinct real numbers contains an ascending subsequence of length $r + 1$ or a descending subsequence of length $s + 1$. *(Hint: Proceed by contradiction. If the sequence does not have an ascending subsequence of length* $r + 1$ *or a decreasing subsequence of length* $s + 1$*, what do you know about the values of all the (I, D) pairs?)*